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Coherent and squeezed states of quantum Heisenberg algebras

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Abstract

Starting from deformed quantum Heisenberg Lie algebras some realizations are given in terms of the usual creation and annihilation operators of the standard harmonic oscillator. Then the associated algebra eigenstates are computed and give rise to new classes of deformed coherent and squeezed states. They are parametrized by deformed algebra parameters and suitable redefinitions of them as paragrassmann numbers. Some properties of these deformed states are also analysed.

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1. Introduction

It is interesting for theoretical and practical reasons to study coherent and squeezed states associated with the quantum Hopf algebras [1–3]. The Hopf algebra structure of a quantum algebra provides us with useful technical elements such as the coproduct, for example. In the case of boson quantum algebras, the special coproduct properties are useful to characterize multi-particle Hamiltonians [4]. For example, in the case of the Poincaré quantum algebra, the coproduct has been brought to bear on the study the fusion of phonons [5]. In general, the concept of deformed quantum Lie algebras found various applications in quantum optics, quantum field theory, quantum statistical mechanics, supersymmetric quantum mechanics and some purely mathematical problems. For instance, in the case of the $su_q(2)$ algebra, it has been found that the $su_q(2)$ effective Hamiltonians reproduce accurately the physical properties of the $su(2) \oplus h(2)$ models [6]. On the other hand, there are some works showing that quantum algebras are connected with paragrassmann algebras [7, 8]. Paragrassmann algebras are relevant in the studies of theories that show the necessity of unusual statistic [9], for instance, the studies of anyons and topological field theories [10, 11].

Now, to associate coherent and squeezed states to a quantum deformed Lie algebra one can use the algebra eigenstates (AES) technique. The AES associated with a real Lie algebra have been defined as the set of eigenstates of an arbitrary complex linear combination of generators of the considered algebra [12]. The AES associated with a quantum real deformed Lie algebra can be defined in a similar way. Indeed, if $A_k(q)$, $k = 1, 2, \ldots, n$ denote the generators of this deformed algebra in a given representation, parametrized by the set of deformation parameters q, then the AES associated with this deformed algebra will be given by the set of solutions of the eigenvalue equation

$$\sum_{k=1}^{n} \alpha_k A_k(q)|\psi\rangle = \lambda|\psi\rangle, \qquad \alpha_k, \lambda \in \mathbb{C}. \tag{1}$$

The purpose of this work is to compute the AES of the deformed quantum Heisenberg Lie algebras [13], obtained by applying the R-matrix methods [1], and find new classes of deformed harmonic oscillator coherent and squeezed states. We will see that these states will be new deformations of the standard coherent and squeezed states of the harmonic oscillator system and we will recover them in the limit when the deformation parameters go to zero. The approach of AES also gives us the possibility of constructing, starting from a deformed algebra, some Hamiltonians of physical systems to which these deformed coherent and squeezed states are associated, similarly as for algebras and superalgebras [14, 15].

It is important to mention that the deformed coherent states obtained by this method differ from the q-deformed coherent states associated with a q-deformed oscillator algebra, which is not a Hopf algebra, constructed by considering either deformed exponential functions, eigenstates of a given deformed annihilation operator, a generalization of the usual form of the standard coherent states, a resolution of the identity technique or a generalized group theoretical techniques [16–18].

The paper is organized as follows. In section 2, a Fock space representation of deformed quantum algebras associated with the Heisenberg algebra $h(2)$ is given. In section 3, we compute the AES associated with these algebras and obtain new classes of deformed coherent and squeezed states that are true deformations of the standard coherent and squeezed states associated with the harmonic oscillator system. These states are parametrized by the deformation parameters which will be considered as real numbers and also as real paragrassmann numbers. In section 4, we compute the product of the dispersions of the position and linear momentum operators of a particle in these states when the parameters of deformation are small. We compare them with the corresponding results obtained in the minimum uncertainty states [14]. Some details of calculations are presented in the appendices A and B. We also give general expressions of these dispersions, in the case where a non-trivial one parameter algebra deformation family is concerned, for all values of the deformation parameter. Finally, we construct a class of η-pseudo Hermitian Hamiltonians [19] to which a subset of these deformed states are the associated coherent states.

2. Deformed quantum Heisenberg algebras in the Fock representation space

We are considering in this work, the deformed Heisenberg quantum algebras obtained by Hussin and Lauzon [13]. They have been obtained using the well-known R-matrix method [1] and are mainly of two types. The first one is formed by the generators A, B, C which satisfy

$$[A, B] = 0, \qquad [B, C] = -\frac{2z}{p^2}(\cosh(pB) - 1), \qquad [A, C] = \frac{1}{p}\sinh(pB). \tag{2}$$

It is denoted by $\mathcal{U}_{z,p}(h(2))$, where p and z are different from zero.

Let us mention that the invertible change of basis

$$\tilde{A} = A, \qquad \tilde{B} = \frac{2}{p}\sinh\left(\frac{pB}{2}\right), \qquad \tilde{C} = \frac{1}{\cosh\left(\frac{pB}{2}\right)}C, \tag{3}$$

leads to the new deformed algebra $\tilde{\mathcal{U}}_{z,0}(h(2))$

$$[\tilde{A}, \tilde{B}] = 0, \qquad [\tilde{B}, \tilde{C}] = -z\tilde{B}^2, \qquad [\tilde{A}, \tilde{C}] = \tilde{B}. \tag{4}$$

This means that we get the same commutation relations as in (2) when p goes to zero. As it has been pointed out by Ballesteros *et al* [20], here the p parameter is superfluous and the families of bialgebras $\mathcal{U}_{z,p}(h(2))$ and $\mathcal{U}_{z,0}(h(2))$ are isomorphic (these families are identified there as of type I_+) on the condition that the coproduct form stands invariant [21].

The second quantum deformation of $h(2)$ is given by

$$[A, B] = [B, C] = 0, \qquad [A, C] = \frac{\mathrm{e}^{pB} - \mathrm{e}^{-qB}}{p+q} \tag{5}$$

and is denoted by $\mathcal{U}_{p,q}(h(2))$, where $p, q \neq 0$. It corresponds to the so-called type II bialgebras in [20]. When $p = q$, we find the quantum Heisenberg algebra obtained in Celeghini *et al* [22] (see also [21]), i.e.,

$$[A, B] = [B, C] = 0, \qquad [A, C] = \frac{1}{p}\sinh(pB). \tag{6}$$

Let us now give a boson realization of these deformed Lie algebras, in terms of the usual creation operator, $a^\dagger$, and annihilation operator, a, associated with the standard quantum harmonic oscillator system. For $\tilde{\mathcal{U}}_{z,0}(h(2))$ given in (4), it is given by

$$\tilde{A} = -a^\dagger, \qquad \tilde{B} = \mathrm{e}^{za^\dagger}, \qquad \tilde{C} = \mathrm{e}^{za^\dagger}a. \tag{7}$$

From (3) and (7), we thus get a realization of $\mathcal{U}_{z,p}(h(2))$ as

$$A = -a^\dagger, \qquad B = \frac{2}{p}\sinh^{-1}\left(\frac{p}{2}\mathrm{e}^{za^\dagger}\right), \qquad C = \mathrm{e}^{za^\dagger}\sqrt{1 + \left(\frac{p}{2}\mathrm{e}^{za^\dagger}\right)^2}\,a. \tag{8}$$

Another realization of $\tilde{\mathcal{U}}_{z,0}(h(2))$ is

$$\tilde{A} = a, \qquad \tilde{B} = \mathrm{e}^{-za}, \qquad \tilde{C} = a^\dagger\,\mathrm{e}^{-za}. \tag{9}$$

We thus get another realization of $\mathcal{U}_{z,p}(h(2))$ as

$$A = a, \qquad B = \frac{2}{p}\sinh^{-1}\left(\frac{p}{2}\mathrm{e}^{-za}\right), \qquad C = a^\dagger\,\mathrm{e}^{-za}\sqrt{1 + \left(\frac{p}{2}\mathrm{e}^{-za}\right)^2}. \tag{10}$$

When z goes to zero, operators (8) become

$$A = -a^\dagger, \qquad B = \frac{2}{p}\sinh^{-1}\left(\frac{p}{2}\right)I, \qquad C = \sqrt{1 + \frac{p^2}{4}}a, \tag{11}$$

while operators (10) become

$$A = a, \qquad B = \frac{2}{p}\sinh^{-1}\left(\frac{p}{2}\right)I, \qquad C = \sqrt{1 + \frac{p^2}{4}}a^\dagger. \tag{12}$$

Operators (11) or (12) thus constitute a realization of deformed Heisenberg algebra (6). When p goes to zero, we regain $h(2)$.

Algebra (5) is clearly isomorphic to $h(2)$ if we introduce

$$\tilde{A} = A, \qquad \tilde{C} = C, \qquad \tilde{B} = \frac{\mathrm{e}^{pB} - \mathrm{e}^{-qB}}{p+q}. \tag{13}$$

So to obtain new class of deformed coherent and squeezed states using the AES method we will deal in the following with $\tilde{\mathcal{U}}_{z,0}(h(2))$ and $\mathcal{U}_{z,p}(h(2))$.

3. AES and deformed coherent and squeezed states

In this section, we compute the AES associated with $\tilde{\mathcal{U}}_{z,0}(h(2))$ and $\mathcal{U}_{z,p}(h(2))$, using the representations obtained in the preceding section. We thus get new classes of deformed coherent and squeezed states associated with the harmonic oscillator system.

3.1. Deformed algebra eigenstates for $\tilde{\mathcal{U}}_{z,0}(h(2))$

We start with $\tilde{\mathcal{U}}_{z,0}(h(2))$ as given by (4) using realizations (7) and (9). The AES are thus defined as the set of solutions of the eigenvalue equation

$$[\alpha_+\tilde{A} + \alpha_0\tilde{B} + \alpha_-\tilde{C}]|\psi\rangle = \alpha|\psi\rangle, \qquad \alpha_-, \alpha_0, \alpha_+, \alpha \in \mathbb{C}. \tag{14}$$

3.1.1. Deformed harmonic oscillator coherent and squeezed states. Let us take first realization (7). Thus, if $\alpha_- \neq 0$, equation (14) can be written in the form

$$\left[\mathrm{e}^{za^\dagger}a + \mu a^\dagger + \nu\,\mathrm{e}^{za^\dagger}\right]|\psi\rangle = \lambda|\psi\rangle, \qquad \mu, \nu, \lambda \in \mathbb{C}. \tag{15}$$

By defining

$$|\psi\rangle = \mathrm{e}^{-\nu a^\dagger}|\varphi\rangle \tag{16}$$

and using $\mathrm{e}^{-\nu a^\dagger}a\,\mathrm{e}^{\nu a^\dagger} = a + \nu$, equation (15) can be reduced to

$$\left[\mathrm{e}^{za^\dagger}a + \mu a^\dagger\right]|\varphi\rangle = \lambda|\varphi\rangle, \qquad \mu, \lambda \in \mathbb{C}. \tag{17}$$

To solve this eigenvalue equation, let us consider the Bargmann space $\mathcal{F}$ of analytic functions $f(\xi)$ $(\xi \in \mathbb{C})$, provided with the scalar product

$$(f_1, f_2) = \int_{\mathbb{C}} \overline{f_1(\xi)}\, f_2(\xi)\,\mathrm{e}^{-\bar{\xi}\xi}\frac{\mathrm{d}\bar{\xi}\,\mathrm{d}\xi}{2\pi\mathrm{i}}, \qquad \forall f_1, f_2 \in \mathcal{F}. \tag{18}$$

It is well known that any function $f \in \mathcal{F}$ can be expressed as a linear combination of orthonormalized functions $u_n(\xi) = \frac{\xi^n}{\sqrt{n!}}$, $n = 0, 1, 2, \ldots$, verifying

$$(u_m, u_n) = \int_{\mathbb{C}} \overline{u_m(\xi)}u_n(\xi)\,\mathrm{e}^{-\bar{\xi}\xi}\frac{\mathrm{d}\bar{\xi}\,\mathrm{d}\xi}{2\pi\mathrm{i}} = \delta_{mn}, \tag{19}$$

that is

$$f(\xi) = \sum_{n=0}^{\infty} c_n u_n(\xi), \tag{20}$$

with

$$c_n = \int_{\mathbb{C}} \overline{u_n(\xi)}\, f(\xi)\,\mathrm{e}^{-\bar{\xi}\xi}\frac{\mathrm{d}\bar{\xi}\,\mathrm{d}\xi}{2\pi\mathrm{i}}. \tag{21}$$

Let us assume a solution of (17) of the type

$$|\varphi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle, \tag{22}$$

where the set of states $\{|n\rangle\}_{n=0}^{\infty}$ forms the basis of the standard Fock oscillator space, verifying the orthogonality relation

$$\langle m|n\rangle = \delta_{mn}. \tag{23}$$

As usual, the action of the operators a and $a^\dagger$ on these states is given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{24}$$

Let us take $|\bar{\xi}\rangle$ with be the standard coherent states associated with the harmonic oscillator system, that is

$$|\bar{\xi}\rangle = \mathrm{e}^{\bar{\xi}a^{\dagger}}|0\rangle = \sum_{n=0}^{\infty} \frac{(\bar{\xi})^n}{\sqrt{n!}}|n\rangle. \tag{25}$$

Then, according to the orthogonality property (23), the projection of $|\varphi\rangle$ on the coherent state $|\bar{\xi}\rangle$ is given by the analytic function

$$\varphi(\xi) = \langle\bar{\xi}|\varphi\rangle = \sum_{n=0}^{\infty} c_n u_n(\xi). \tag{26}$$

The action of the operators $a^{\dagger}$ and a in this representation corresponds to

$$\langle\bar{\xi}|a^{\dagger}|\varphi\rangle = \xi\varphi(\xi), \qquad \langle\bar{\xi}|a|\varphi\rangle = \frac{\mathrm{d}\varphi}{\mathrm{d}\xi}(\xi) \tag{27}$$

respectively. Thus, by projecting both sides of the eigenvalue equation (17) on the coherent states $|\bar{\xi}\rangle$ and then using (27), we can write it as

$$\left(\mathrm{e}^{z\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} + \mu\xi\right)\varphi(\xi) = \lambda\varphi(\xi). \tag{28}$$

The general solution of this differential equation is given by

$$\varphi(\xi) = C_0(\lambda, \mu, z)\exp\left(\sum_{k=0}^{\infty} \frac{(-z\xi)^k}{(k+1)!}\left(\lambda\xi - \frac{k+1}{k+2}\mu\xi^2\right)\right), \tag{29}$$

where C_0 is an arbitrary constant which can be fixed from the normalization condition

$$(\varphi, \varphi) = \int_{\mathbb{C}} \overline{\varphi(\xi)}\varphi(\xi)\,\mathrm{e}^{-\bar{\xi}\xi}\frac{\mathrm{d}\bar{\xi}\,\mathrm{d}\xi}{2\pi\mathrm{i}} = 1. \tag{30}$$

Let us note that in the particular limit when z goes to zero, solution (29) becomes the symbol for the squeezed states [23] associated with the standard harmonic oscillator, that is

$$\varphi(\xi) = C_0(\lambda, \mu, 0)\exp\left(\lambda\xi - \frac{\mu}{2}\xi^2\right). \tag{31}$$

This quantity is normalizable only if $|\mu| < 1$ [24].

When $z \neq 0$, solution (29) can be written in the form

$$\varphi(\xi) = C_0(\lambda, \mu, z)\exp\left(\frac{\lambda}{z} - \frac{\mu}{z^2}\right)\exp\left(\mathrm{e}^{-z\xi}\frac{(\mu - \lambda z + \mu z\xi)}{z^2}\right). \tag{32}$$

Going back to expression (26), we get the coefficients c_n, $n = 0, 1, \ldots,$ as

$$\begin{aligned} c_n = \int_{\mathbb{C}} \overline{u_n(\xi)}\varphi(\xi)\,\mathrm{e}^{-\bar{\xi}\xi}\frac{\mathrm{d}\bar{\xi}\,\mathrm{d}\xi}{2\pi\mathrm{i}} &= C_0(\lambda, \mu, z)\exp\left(\frac{\lambda}{z} - \frac{\mu}{z^2}\right) \\ &\times \int_{\mathbb{C}} \frac{\bar{\xi}^n}{\sqrt{n!}}\exp\left(\mathrm{e}^{-z\xi}\frac{(\mu - \lambda z + \mu z\xi)}{z^2}\right)\mathrm{e}^{-\bar{\xi}\xi}\frac{\mathrm{d}\bar{\xi}\,\mathrm{d}\xi}{2\pi\mathrm{i}}. \end{aligned} \tag{33}$$

By using the polar change of variables $\xi = \rho\,\mathrm{e}^{\mathrm{i}\vartheta}$, this last equation can be written in the form

$$\begin{aligned} c_n &= C_0(\lambda, \mu, z)\exp\left(\frac{\lambda}{z} - \frac{\mu}{z^2}\right) \\ &\times \int_0^{\infty}\int_0^{2\pi} \frac{\rho^{n+1}\,\mathrm{e}^{-\rho^2}}{\sqrt{n!}}\,\mathrm{e}^{-\mathrm{i}n\vartheta}\exp\left(\frac{\mathrm{e}^{-z\rho\,\mathrm{e}^{\mathrm{i}\vartheta}}}{z^2}(\mu - \lambda z + \mu z\rho\,\mathrm{e}^{\mathrm{i}\vartheta})\right)\frac{\mathrm{d}\rho\,\mathrm{d}\vartheta}{\pi}. \end{aligned} \tag{34}$$

Let us write the exponential factor in the form

$$\exp\left(\frac{\mathrm{e}^{-z\rho\,\mathrm{e}^{\mathrm{i}\vartheta}}}{z^2}(\mu-\lambda z+\mu z\rho\,\mathrm{e}^{\mathrm{i}\vartheta})\right)=\sum_{k=0}^{\infty}\frac{\exp(-zk\rho\,\mathrm{e}^{\mathrm{i}\vartheta})}{k!}\left(\frac{\mu-\lambda z+uz\rho\,\mathrm{e}^{\mathrm{i}\vartheta}}{z^2}\right)^k$$

$$=\sum_{k,l=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m}\rho^{l+m}\,\mathrm{e}^{\mathrm{i}(l+m)\vartheta}\frac{(-zk)^l(\mu z)^m(\mu-\lambda z)^{k-m}}{k!l!z^{2k}} \tag{35}$$

to get

$$c_n=C_0(\lambda,\mu,z)\exp\left(\frac{\lambda}{z}-\frac{\mu}{z^2}\right)\sum_{k,l=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m}\frac{(-zk)^l(\mu z)^m(\mu-\lambda z)^{k-m}}{\sqrt{n!}k!l!z^{2k}}$$
$$\times\left(\int_0^{\infty}\rho^{m+l+n+1}\,\mathrm{e}^{-\rho^2}\,\mathrm{d}\rho\right)\left(\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(l+m-n)\vartheta}\frac{\mathrm{d}\vartheta}{\pi}\right). \tag{36}$$

Using the known results

$$\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(l+m-n)\vartheta}\frac{\mathrm{d}\vartheta}{\pi}=2\delta_{l+m-n,0}, \tag{37}$$

$$\int_0^{\infty}\rho^{m+l+n+1}\,\mathrm{e}^{-\rho^2}\,\mathrm{d}\rho=\frac{1}{2}\Gamma\left(\frac{m+l+n}{2}+1\right), \tag{38}$$

and performing the sum over the index l, the expression for the coefficients c_n reduces to

$$c_n=C_0(\lambda,\mu,z)\exp\left(\frac{\lambda}{z}-\frac{\mu}{z^2}\right)\frac{z^n}{\sqrt{n!}}\sum_{k=0}^{\infty}\sum_{m=0}^{k_<}\binom{n}{m}\frac{(-k)^{n-m}}{(k-m)!}\left(\frac{\mu}{z^2}\right)^m\left(\frac{\mu}{z^2}-\frac{\lambda}{z}\right)^{k-m}, \tag{39}$$

where $k_<$ denotes the minimum between k and n. This last expression can be written in the form

$$c_n=C_0(\lambda,\mu,z)\frac{z^n}{\sqrt{n!}}\sum_{m=0}^{n}\sum_{j=0}^{n-m}\binom{n}{m}(-1)^{n-m}\upsilon_{mj}\left(\frac{\mu}{z^2}\right)^m\left(\frac{\mu}{z^2}-\frac{\lambda}{z}\right)^j, \tag{40}$$

where the coefficients υ_{mj} are obtained from

$$\frac{k^{n-m}}{(k-m)!}=\sum_{j=0}^{n-m}\frac{\upsilon_{mj}}{(k-m-j)!}. \tag{41}$$

Thus the coefficients $c_n, n=1,2,\ldots,$ represent polynomials of degree $n-1$ in the z variable. For example, $c_1=\lambda C_0$,

$$c_2=C_0\sqrt{2!}\left[\left(\frac{\lambda^2}{2!}-\frac{\mu}{2}\right)-\frac{\lambda}{2}z\right],$$
$$c_3=C_0\sqrt{3!}\left[\left(\frac{\lambda^3}{3!}-\frac{\mu\lambda}{2}\right)+\left(\frac{\mu}{3}-\frac{\lambda^2}{2}\right)z+\frac{\lambda}{6}z^2\right]. \tag{42}$$

The normalization constant C_0 can now be computed. Indeed, inserting (40) into (26) and the resulting expression into the normalization condition (30), using the orthogonality relation (19), we get

$$C_0(\lambda,\mu,z)=\left[\sum_{n=0}^{\infty}\frac{z^{2n}}{n!}\sum_{m=0}^{n}\sum_{r=0}^{n}\sum_{j=0}^{n-m}\sum_{l=0}^{n-r}\binom{n}{m}\binom{n}{r}(-1)^{m+r}\upsilon_{mj}\upsilon_{rl}\right.$$
$$\left.\times\left(\frac{\mu}{z^2}\right)^m\left(\frac{\bar{\mu}}{z^2}\right)^r\left(\frac{\mu}{z^2}-\frac{\lambda}{z}\right)^j\left(\frac{\bar{\mu}}{z^2}-\frac{\bar{\lambda}}{z}\right)^l\right]^{-\frac{1}{2}}, \tag{43}$$

which has been chosen real. The convergence of these series is not easy to determine. In the case where $z = 0$, as we have already mentioned, the series $\sum_{n=0}^{\infty} |c_n|^2$ converges for all λ provided that $|\mu| < 1$. In the case $\mu = 0$, this series becomes

$$\sum_{n=0}^{\infty} |c_n|^2 = |C_0(\lambda, z)|^2 \exp\left(\frac{\lambda}{z}\right) \sum_{n=0}^{\infty} \frac{\left(-\frac{\lambda}{z}\right)^n}{n!} \exp\left(-\frac{\bar{\lambda}}{z} \sum_{k=1}^{\infty} \frac{(z^2 n)^k}{k!}\right). \tag{44}$$

It converges for all $z > 0$ provided that the phase θ in $\lambda = \beta\,\mathrm{e}^{\mathrm{i}\theta}$ satisfies $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$, whereas for all $z < 0$, it converges if $\frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}$.

Finally, we can show that the normalized algebra eigenstates $|\varphi\rangle$, solving (17) , can be expressed in terms of a deformed squeezed operator acting on the ground state of the standard harmonic oscillator, that is

$$|\varphi\rangle = C_0(\lambda, \mu, z) \exp\left(\sum_{k=0}^{\infty} \frac{(-za^\dagger)^k}{(k+1)!} \left(\lambda a^\dagger - \frac{k+1}{k+2}\mu (a^\dagger)^2\right)\right) |0\rangle. \tag{45}$$

Also, combining this last equation with equation (16), we get the algebra eigenstates solving (15) to be the deformed coherent states

$$|\psi\rangle = N_0(\lambda, \mu, \nu, z) \exp\left(\sum_{k=0}^{\infty} \frac{(-za^\dagger)^k}{(k+1)!} \left(\lambda a^\dagger - \frac{k+1}{k+2}\mu (a^\dagger)^2\right)\right) \mathrm{e}^{-\nu a^\dagger} |0\rangle, \tag{46}$$

where $N_0(\lambda, \mu, \nu, z)$ is a normalization constant which can be computed in the same way as $C_0(\lambda, \mu, z)$.

3.1.2. Perturbed squeezed states. Let us now assume that z is a small perturbation parameter of order $k_0 - 1$, where k_0 is an integer greater or equal to 2. From (45), neglecting the terms containing the power of z greater than $k_0 - 1$, we can write

$$\begin{aligned} |\varphi\rangle \approx C_0(\lambda, \mu, z, k_0) \Bigg[1 + \sum_{k=1}^{k_0-1} \frac{(-za^\dagger)^k}{(k+1)!} \left(\lambda a^\dagger - \frac{k+1}{k+2}\mu (a^\dagger)^2\right) \\ + \cdots + \frac{1}{(k_0-1)!} \left(\frac{-za^\dagger}{2!} \left(\lambda a^\dagger - \frac{2}{3}\mu (a^\dagger)^2\right)\right)^{k_0-1} \Bigg] \exp\left(\lambda a^\dagger - \frac{\mu}{2}(a^\dagger)^2\right) |0\rangle. \end{aligned} \tag{47}$$

These states can be normalized in the standard form. For instance, when $k_0 = 2$, $\mu = \delta\,\mathrm{e}^{\mathrm{i}\phi}$, $\lambda = \beta\,\mathrm{e}^{\mathrm{i}\theta}$, where ϕ and θ are real phases, $0 \leqslant \delta < 1$, and $\beta \geqslant 0$, a normalized version of the deformed squeezed states (47) is given by

$$|\varphi\rangle \approx \Omega(\delta, \phi, \beta, \theta) \left[1 + z\left(\frac{\delta\,\mathrm{e}^{\mathrm{i}\phi}}{3}(a^\dagger)^3 - \frac{\beta\,\mathrm{e}^{\mathrm{i}\theta}}{2}(a^\dagger)^2\right)\right] S(-\arctan(\delta)\,\mathrm{e}^{\mathrm{i}\phi}) D\left(\frac{\beta\,\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}}\right) |0\rangle, \tag{48}$$

where

$$\begin{aligned} \Omega(\delta, \phi, \beta, \theta) = 1 + \frac{z\beta}{2(1-\delta^2)^2} \Bigg[\left(2\delta^2 + \beta^2 \left(\frac{1+\delta^2}{1-\delta^2}\right)\right) \cos\theta \\ - \delta \left(1 + \delta^2 + \frac{2\beta^2}{1-\delta^2}\right) \cos(\phi - \theta) \\ + \delta^2 \beta^2 \left(1 + \frac{2\delta^2}{3(1-\delta^2)}\right) \cos(2\phi - 3\theta) - \frac{2\delta\beta^2}{3(1-\delta^2)} \cos(\phi - 3\theta) \Bigg]. \end{aligned} \tag{49}$$

Here $S(\chi) = \exp\left[-\left(\chi\frac{(a^\dagger)^2}{2} - \bar{\chi}\frac{a^2}{2}\right)\right]$ is the standard unitary squeezed operator [25] and $D(\lambda) = \exp(\lambda a^\dagger - \bar{\lambda}a)$ the standard displacement operator [26].

3.1.3. Deformed squeezed and coherent states parametrized by paragrassmann numbers. Let us now use realization (9) of $\tilde{\mathcal{U}}_{z,0}(h(2))$. In the case $\alpha_+ \neq 0$, equation (14) can be now written in the form

$$[a + \mu a^\dagger \,\mathrm{e}^{-za} + \nu\,\mathrm{e}^{-za}]|\psi\rangle = \lambda|\psi\rangle, \qquad \mu, \nu, \lambda \in \mathbb{C}. \tag{50}$$

There are two types of equations to solve. The first type is obtained when $\mu \neq 0$ and $\nu \neq 0$. We can take

$$|\psi\rangle = \exp\left(\frac{\nu}{\mu}a\right)|\varphi\rangle \tag{51}$$

and use the relation, $\exp\left(-\frac{\nu}{\mu}a\right)a^\dagger \exp\left(\frac{\nu}{\mu}a\right) = a^\dagger - \frac{\nu}{\mu}$, to reduce (50) to the form

$$[a + \mu a^\dagger \,\mathrm{e}^{-za}]|\varphi\rangle = \lambda|\varphi\rangle, \qquad \mu, \lambda \in \mathbb{C}. \tag{52}$$

If $\nu = 0$ and $\mu \neq 0$, we see from (50) that the same type of eigenvalue equation must be solved. The second type is obtained when $\mu = 0$. The eigenvalue equation is

$$[a + \nu\,\mathrm{e}^{-za}]|\psi\rangle = \lambda|\psi\rangle, \qquad \nu, \lambda \in \mathbb{C}. \tag{53}$$

We begin with the resolution of equation (52). Let us assume $|\varphi\rangle$ to be again a solution of type (22). Thus, proceeding as in the preceding section, the eigenvalue equation satisfied by the symbol $\varphi(\xi)$, in the Bargmann representation, is given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}\xi} + \mu\xi\,\mathrm{e}^{-z\frac{\mathrm{d}}{\mathrm{d}\xi}}\right)\varphi(\xi) = \lambda\varphi(\xi), \qquad \mu, \lambda \in \mathbb{C}. \tag{54}$$

To solve this equation, let us assume that z is a real paragrassmann number [8, 9], that is $z^{k_0} = 0$, for some integer $k_0 \geqslant 1$. A detailed procedure of resolution of this equation is given in appendix A. Let us note that the case $k_0 = 1$, i.e., $z = 0$, is somewhat trivial since the eigenfunctions $\varphi(\xi)$ solving (54) are given by the standard squeezed symbol (31). When $k_0 = 2$, or $z^2 = 0$, i.e., when z is a odd Grassmann number [27, 28], the eigenvalue equation (54) becomes

$$\left((1 - \mu z\xi)\frac{\mathrm{d}}{\mathrm{d}\xi} + \mu\xi\right)\varphi(\xi) = \lambda\varphi(\xi), \qquad \mu, \lambda \in \mathbb{C}. \tag{55}$$

There are two independent solutions (see appendix A). The normalizable solution of this eigenvalue equation is given by the deformed squeezed symbol

$$\varphi(\lambda, \mu, z)(\xi) = C_0(\lambda, \mu, z)\left[1 + z\mu\left(\lambda\frac{\xi^2}{2} - \mu\frac{\xi^3}{3}\right)\right]\exp\left(\lambda\xi - \frac{\mu}{2}\xi^2\right). \tag{56}$$

A normalized version of these states, in the Fock space representation, is given by

$$\begin{aligned}|\varphi\rangle = \tilde{\Omega}(\delta, \phi, \beta, \theta)&\left[1 + z\delta\left(\frac{\delta\,\mathrm{e}^{2\mathrm{i}\phi}}{3}(a^\dagger)^3 - \frac{\beta\,\mathrm{e}^{\mathrm{i}(\theta+\phi)}}{2}(a^\dagger)^2\right)\right]\\ &\times S(-\arctan(\delta)\,\mathrm{e}^{\mathrm{i}\phi})D\left(\frac{\beta\,\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}}\right)|0\rangle,\end{aligned} \tag{57}$$

where λ and μ have been chosen as in the preceding subsection and

$$\tilde{\Omega}(\delta, \phi, \beta, \theta) = 1 - \frac{z\delta\beta}{2(1-\delta^2)^2}\left[\left(2\delta^2 + \beta^2\left(\frac{1+\delta^2}{1-\delta^2}\right)\right)\cos(\theta - \phi)\right.$$

$$- \delta \left(1 + \delta^2 + \frac{2\beta^2}{1-\delta^2}\right) \cos\theta + \delta^2\beta^2 \left(1 + \frac{2\delta^2}{3(1-\delta^2)}\right) \cos(\phi - 3\theta)$$
$$\left. - \frac{2\delta\beta^2}{3(1-\delta^2)} \cos(2\phi - 3\theta)\right]. \tag{58}$$

When $k_0 = 3$, or $z^3 = 0$, the eigenvalue equation (54) becomes the second-order differential equation

$$\left(\frac{1}{2}\mu z^2 \xi \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + (1 - \mu z \xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\varphi(\xi) = (\lambda - \mu\xi)\varphi(\xi), \qquad \mu, \lambda, \in \mathbb{C}. \tag{59}$$

According to the results obtained in appendix A, the general solution of this equation can be expanded in the form

$$\varphi(\xi) = \varphi_0(\xi) + z\varphi_1(\xi) + z^2\varphi_2(\xi), \tag{60}$$

with

$$\varphi_0(\xi) = C_0 \exp\left(\lambda\xi - \mu\frac{\xi^2}{2}\right), \tag{61}$$

$$\varphi_1(\xi) = \left[\mu\left(\lambda\frac{\xi^2}{2} - \mu\frac{\xi^3}{3}\right)C_0 + C_1\right]\exp\left(\lambda\xi - \mu\frac{\xi^2}{2}\right), \tag{62}$$

$$\varphi_2(\xi) = \left[\left(\mu(\mu - \lambda^2)\frac{\xi^2}{4} + \frac{2}{3}\mu^2\lambda\xi^3 + \mu^2(\lambda^2 - 3\mu)\frac{\xi^4}{8} - \lambda\mu^3\frac{\xi^5}{6} + \mu^4\frac{\xi^6}{18}\right)C_0\right.$$
$$\left. + \mu\left(\lambda\frac{\xi^2}{2} - \mu\frac{\xi^3}{3}\right)C_1 + C_2\right]\exp\left(\lambda\xi - \mu\frac{\xi^2}{2}\right), \tag{63}$$

where C_0, C_1 and C_2 are arbitrary integration constants. Three independent solutions may thus be obtained. The first one is obtained by taking $C_1 = C_2 = 0$. We get

$$\varphi(\xi) = C_0\left[1 + z\mu\left(\lambda\frac{\xi^2}{2} - \mu\frac{\xi^3}{3}\right) + z^2\left(\mu(\mu - \lambda^2)\frac{\xi^2}{4} + \frac{2}{3}\mu^2\lambda\xi^3\right.\right.$$
$$\left.\left. + \mu^2(\lambda^2 - 3\mu)\frac{\xi^4}{8} - \lambda\mu^3\frac{\xi^5}{6} + \mu^4\frac{\xi^6}{18}\right)\right]\exp\left(\lambda\xi - \mu\frac{\xi^2}{2}\right) \tag{64}$$

$$= C_0 \exp\left[z\mu\left(\lambda\frac{\xi^2}{2} - \mu\frac{\xi^3}{3}\right) + z^2 f(\xi)\right]\exp\left(\lambda\xi - \mu\frac{\xi^2}{2}\right), \tag{65}$$

where

$$f(\xi) = \left(\mu(\mu - \lambda^2)\frac{\xi^2}{4} + \frac{2}{3}\mu^2\lambda\xi^3 - 3\mu^3\frac{\xi^4}{8}\right). \tag{66}$$

This solution can be normalized and represents a second-order paragrassmann deformation of squeezed states associated with the standard harmonic oscillator.

The other independent solutions are given respectively by

$$\varphi(\xi) = C_1 z\left[1 + z\mu\left(\lambda\frac{\xi^2}{2} - \mu\frac{\xi^3}{3}\right)\right]\exp\left(\lambda\xi - \mu\frac{\xi^2}{2}\right) \tag{67}$$

and

$$\varphi(\xi) = C_2 z^2 \exp\left(\lambda\xi - \mu\frac{\xi^2}{2}\right). \tag{68}$$

These solutions cannot be normalized since $z^k, k = 1, 2,$ are not invertible paragrassmann numbers and $z^k = 0, k = 3, 4, \ldots$.

The higher order paragrassmann deformations of the squeezed states associated with the standard harmonic oscillator can be obtained following a similar procedure (see appendix A).

In the case of eigenvalue equation (53), the differential equation to solve is given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}\xi} + \nu\,\mathrm{e}^{-z\frac{\mathrm{d}}{\mathrm{d}\xi}}\right)\varphi(\xi) = \lambda\varphi(\xi), \qquad \nu, \lambda, \in \mathbb{C}. \tag{69}$$

Proceedings as before and considering the results of appendix A, the normalizable solutions of this last equation, when $k_0 = 1, 2, 3$, are given respectively by the deformed coherent symbols

$$\varphi^{(1)}(\xi) = C_0 \exp((\lambda - \nu)\xi), \tag{70}$$

$$\varphi^{(2)}(\xi) = C_0\,[1 + z(\lambda - \nu)\nu\xi]\exp((\lambda - \nu)\xi) \tag{71}$$

and

$$\varphi^{(3)}(\xi) = C_0\left\{1 + z(\lambda - \nu)\nu\xi + z^2\left[\left(\frac{\lambda^2\nu}{2} + 2\lambda\nu^2 - \frac{3\nu^3}{2}\right)\xi + \left(\frac{\lambda^2\nu^2}{2} - \lambda\nu^3 + \frac{\nu^4}{2}\right)\xi^2\right]\right\}\exp((\lambda - \nu)\xi). \tag{72}$$

These solutions can be normalized and represent zero-, first- and second-order paragrassmann deformations, respectively, of coherent states associated with the standard harmonic oscillator. For higher values of k_0, we must proceed as in appendix A.

3.2. *Deformed algebra eigenstates for $\mathcal{U}_{z,p}(h(2))$*

It is interesting to compute the AES associated with $\mathcal{U}_{z,p}(h(2)), z, p \neq 0$, and compare it with those associated with $\mathcal{U}_{z,0}(h(2))$. As we have noted in section 2, these quantum algebras are isomorphic in the sense that there is a nonlinear change of basis transforming one to the other. In general, the existence of this isomorphism does not imply the existence of an internal homomorphism at the AES level. Indeed, by definition, the eigenvalue equation determining the set of AES deals with an arbitrary linear combination of the deformed algebra generators, then from the inverses of transformations (3) and the solvable structure of the commutation relations (4), it is impossible to find an internal homomorphism, at the AES level, transforming the eigenvalue equation with $z, p \neq 0$ to the eigenvalue equation with $z \neq 0, p = 0$.

To see that, in this section, we consider the two parameters deformed algebra $\mathcal{U}_{z,p}(h(2))$ as given by (2), and compute the AES using the particular realization (8). More precisely, we have to solve the eigenvalue equation

$$\left[\mathrm{e}^{za^\dagger}\sqrt{1 + \left(\frac{p}{2}\,\mathrm{e}^{za^\dagger}\right)^2}\,a + \mu a^\dagger + \frac{2\nu}{p}\sinh^{-1}\left(\frac{p}{2}\,\mathrm{e}^{za^\dagger}\right)\right]|\psi\rangle = \lambda|\psi\rangle, \qquad \mu, \nu, \lambda \in \mathbb{C}. \tag{73}$$

In the Bargmann representation, this equation becomes the first-order differential equation

$$\left[\mathrm{e}^{z\xi}\sqrt{1 + \left(\frac{p}{2}\,\mathrm{e}^{z\xi}\right)^2}\,\frac{\mathrm{d}}{\mathrm{d}\xi} + \mu\xi + \frac{2\nu}{p}\sinh^{-1}\left(\frac{p}{2}\,\mathrm{e}^{z\xi}\right)\right]\psi(\xi) = \lambda\psi(\xi), \qquad \mu, \nu, \lambda \in \mathbb{C}. \tag{74}$$

When $z = 0$, we easily get the standard squeezed symbols

$$\psi_{0,p}(\xi) = C_0(p, \lambda, \mu, \nu) \exp\left[\left(\lambda - \frac{2\nu}{p}\sinh^{-1}(p/2)\right)\xi - \mu\frac{\xi^2}{2}\right]. \tag{75}$$

These symbols correspond with the Bargmann representation of the AES associated with the deformed quantum Heisenberg algebra realization (11). Moreover, when p goes to zero, these symbols become the standard squeezed symbols associated with $h(2)$.

When $z \neq 0$, making the change of variable $\zeta = e^{z\xi}$, rearranging the terms and using the method of characteristics curves to separate the differentials, we get

$$\frac{d\psi}{\psi}(\zeta) = \frac{\left[\lambda - \frac{\mu}{z}\ln\zeta - \frac{2\nu}{p}\sinh^{-1}\left(\frac{p\zeta}{2}\right)\right]}{z\zeta^2\sqrt{1 + \frac{p^2\zeta^2}{4}}}\, d\zeta. \tag{76}$$

Integrating both sides of this equation and then exponentiating, we get

$$\psi_{z,p}(\zeta) = C_0(\lambda, \mu, \nu; z, p) \exp\left[\frac{\sqrt{1 + \frac{p^2\zeta^2}{4}}}{z^2\zeta}\left((1 + \ln\zeta)\mu - \lambda z + \frac{2\nu z}{p}\sinh^{-1}\left(\frac{p\zeta}{2}\right)\right) - \frac{\mu p}{2z^2}\sinh^{-1}\left(\frac{p\zeta}{2}\right) - \frac{\nu}{z}\ln\zeta\right]. \tag{77}$$

This result includes those obtained for (15) when p goes to zero. Moreover, when we set also $\nu = 0$, we regain (32).

3.2.1. Perturbed two parameters deformation coherent and squeezed states. Up to first order of approximation in z and p^2, the deformed symbol (77) writes

$$\psi_{z,p}(\xi) \approx \tilde{C}_0(\lambda, \mu, \nu; z, p)\left[1 + z\left(\frac{\mu\xi^3}{3} - \frac{\lambda\xi^2}{2}\right) + \frac{p^2}{4}\left(\frac{\mu\xi^2}{4} - \left(\frac{\lambda}{2} - \frac{\nu}{3}\right)\xi\right)\right] \exp\left((\lambda - \nu)\xi - \frac{1}{2}\mu\xi^2\right). \tag{78}$$

In the case $\mu = \delta\, e^{i\phi}$, $\lambda = \beta\, e^{i\theta}$ and $\nu = -\gamma\, e^{i\eta}$, where $\gamma \geqslant 0$, a normalized version of these states, in the Fock representation, is given by

$$|\psi\rangle \approx \tilde{\Omega}(\delta, \phi, \beta, \theta, \gamma, \eta)\left\{1 + \left[z\left(\frac{\delta\, e^{i\phi}}{3}(a^\dagger)^3 - \frac{\beta\, e^{i\theta}}{2}(a^\dagger)^2\right)\right] + \frac{p^2}{4}\left[\frac{\delta\, e^{i\phi}}{4}(a^\dagger)^2 - \left(\frac{\beta\, e^{i\theta}}{2} + \frac{\gamma\, e^{i\eta}}{3}\right)a^\dagger\right]\right\} \times S(-\arctan(\delta)\, e^{i\phi}) D\left(\frac{\tilde{\beta}\, e^{i\tilde{\theta}}}{\sqrt{1 - \delta^2}}\right)|0\rangle, \tag{79}$$

where

$$\tilde{\Omega}(\delta, \phi, \beta, \theta, \gamma, \eta) = 1 + \frac{z}{2(1 - \delta^2)^2}\left\{\tilde{\beta}\left[\left(2\delta^2 + \tilde{\beta}^2\left(\frac{1 + \delta^2}{1 - \delta^2}\right)\right)\cos\tilde{\theta} - \delta\left(1 + \delta^2 + \frac{2\tilde{\beta}^2}{1 - \delta^2}\right)\cos(\phi - \tilde{\theta})\right.\right.$$

$$+\delta^2\tilde{\beta}^2\left(1+\frac{2\delta^2}{3(1-\delta^2)}\right)\cos(2\phi-3\tilde{\theta})-\frac{2\delta\tilde{\beta}^2}{3(1-\delta^2)}\cos(\phi-3\tilde{\theta})\Bigg]$$

$$-\gamma\Bigg[\tilde{\beta}^2\cos(\eta-2\tilde{\theta})-\delta(2\tilde{\beta}^2+1-\delta^2)\cos(\eta-\tilde{\theta})$$

$$+\delta^2\tilde{\beta}^2\cos(2\phi-\eta-2\tilde{\theta})\Bigg]\Bigg]\Bigg\}-\frac{p^2}{16(1-\delta^2)^2}\Bigg\{\delta\tilde{\beta}^2(3\cos(\phi-2\tilde{\theta})$$

$$+\frac{2\gamma}{3}\tilde{\beta}(1-\delta^2)(\cos(\eta-\tilde{\theta})+\delta\cos(\phi-\eta-\tilde{\theta}))-2\tilde{\beta}^2-\delta^2+\delta^4\Bigg\}, \tag{80}$$

where

$$\tilde{\beta}=\sqrt{\beta^2+\gamma^2+2\beta\gamma\cos(\eta-\theta)}, \qquad \tilde{\theta}=\tan^{-1}\left(\frac{\beta\sin\theta+\gamma\sin\eta}{\beta\cos\theta+\gamma\cos\eta}\right). \tag{81}$$

We note that, in the case $\gamma = 0$ and $p = 0$, these normalized states become the normalized states given in equation (48).

4. Some properties of the deformed states

In this section, we will give some properties of the deformed states found in the preceding section. From Fock space representation, we will deduce the physical quantities X and P, representing the position and linear momentum of a particle, respectively, and compute the corresponding dispersions in both the perturbed deformed states associated with $\mathcal{U}_{z,p}(h(2))$ and the deformed states associated with $\tilde{\mathcal{U}}_{z,0}(h(2))$. We will also connect the last states with an η-pseudo Hermitian Halmiltonian [19].

4.1. Squeezing properties

First, let us consider the squeezing properties of X and P. In the Fock space representation, these quantities are given by the Hermitian operators (we have assumed that the mass, angular frequency and Planck's constant are all equal to 1)

$$X=\frac{(a+a^\dagger)}{\sqrt{2}}, \qquad P=\mathrm{i}\frac{(a^\dagger-a)}{\sqrt{2}}. \tag{82}$$

They verify the canonical commutation relation

$$[X,P]=\mathrm{i}I. \tag{83}$$

The dispersion of these quantities, computed on a specific normalized particle state $|\psi\rangle$, is defined as

$$(\Delta X)^2=\langle\psi|X^2|\psi\rangle-(\langle\psi|X|\psi\rangle)^2 \tag{84}$$

and

$$(\Delta P)^2=\langle\psi|P^2|\psi\rangle-(\langle\psi|P|\psi\rangle)^2. \tag{85}$$

The product of these dispersions satisfies the Schrödinger–Robertson uncertainty relation (SRUR) [29, 30]

$$(\Delta X)^2(\Delta P)^2\geqslant\tfrac{1}{4}(\langle I\rangle^2+\langle F\rangle^2)=\tfrac{1}{2}(1+\langle F\rangle^2), \tag{86}$$

where F is the anti-commutator $F = \{X - \langle X\rangle I, P - \langle P\rangle I\}$. The mean value of F is a correlation measure between X and P. When $\langle F\rangle = 0$, we regain the standard Heisenberg uncertainty principle.

The minimum uncertainty states (MUS) are states that satisfy the equality in (86). They are called coherent states when the dispersions of both X and P are the same and squeezed states when these dispersions are different to each other. The states for which the dispersion of X is greater than that of P are called X squeezed whereas the states for which the dispersion of P is greater than that of X are called P squeezed.

We are interested to compute the dispersions of X and P, in the deformed squeezed states (79), when $\nu = 0$, or $\gamma = 0$. More precisely, we want to study the effect of the deformation parameters on the squeezed properties of these quantities. As we have seen, when z and p go to zero, states (79) become the standard harmonic oscillator squeezed states. In such a case, we know that the dispersions of X and P are independent of $\lambda = \beta\,\mathrm{e}^{\mathrm{i}\theta}$, and given by [14]

$$(\Delta X)_0{}^2 = \frac{1 - 2\delta\cos\phi + \delta^2}{2(1-\delta^2)} \quad \text{and} \quad (\Delta P)_0{}^2 = \frac{1 + 2\delta\cos\phi + \delta^2}{2(1-\delta^2)}. \tag{87}$$

All these states are MUS, that is, they satisfy the equality in (86).

When $\gamma = 0$, the square of the mean value of X, in states (79), to first order of approximation in z and p^2, is given by

$$\begin{aligned}\langle\psi|X|\psi\rangle^2 \approx 2(\mathrm{Re}\,\Gamma_{01})\,\mathrm{Re}\Bigg\{&(1 + 4\epsilon(z,p))\Gamma_{01}\\ &+ 2z\left(\frac{\delta\,\mathrm{e}^{-\mathrm{i}\phi}}{3}\Gamma_{04} - \frac{\beta\,\mathrm{e}^{-\mathrm{i}\theta}}{2}\Gamma_{03} + \frac{\delta\,\mathrm{e}^{\mathrm{i}\phi}}{3}\Lambda_{13} - \frac{\beta\,\mathrm{e}^{\mathrm{i}\theta}}{2}\Lambda_{12}\right)\\ &+ \frac{p^2}{2}\left(\frac{\delta\,\mathrm{e}^{-\mathrm{i}\phi}}{4}\Gamma_{03} - \frac{\beta\,\mathrm{e}^{-\mathrm{i}\theta}}{2}\Gamma_{02} + \frac{\delta\,\mathrm{e}^{\mathrm{i}\phi}}{4}\Lambda_{12} - \frac{\beta\,\mathrm{e}^{\mathrm{i}\theta}}{2}\Lambda_{11}\right)\Bigg\},\end{aligned} \tag{88}$$

where $\epsilon(z,p) = \tilde{\Omega}(\delta,\phi,\beta,\theta,0,0) - 1$ and Γ_{kl} and Λ_{kl}, $k, l = 1, 2, \ldots$, are matrix elements defined in appendix B. According to (82), we have the same expression for the square of the mean value of P, but taking the imaginary part in place of the real part.

On the other hand, the mean value of X^2 in (79), to first order of approximation in z and p^2, is given by

$$\begin{aligned}\langle\psi|X^2|\psi\rangle \approx \frac{1}{2} &+ (1 + 2\epsilon(z,p))(\Gamma_{11} + \mathrm{Re}\,\Gamma_{02})\\ &+ z\,\mathrm{Re}\left(\frac{\delta\,\mathrm{e}^{-\mathrm{i}\phi}}{3}\Gamma_{05} - \frac{\beta\,\mathrm{e}^{-\mathrm{i}\theta}}{2}\Gamma_{04} + \frac{\delta\,\mathrm{e}^{\mathrm{i}\phi}}{3}\Lambda_{23} - \frac{\beta\,\mathrm{e}^{\mathrm{i}\theta}}{2}\Lambda_{22}\right)\\ &+ \frac{p^2}{4}\,\mathrm{Re}\left(\frac{\delta\,\mathrm{e}^{-\mathrm{i}\phi}}{4}\Gamma_{04} - \frac{\beta\,\mathrm{e}^{-\mathrm{i}\theta}}{2}\Gamma_{03} + \frac{\delta\,\mathrm{e}^{\mathrm{i}\phi}}{4}\Lambda_{22} - \frac{\beta\,\mathrm{e}^{\mathrm{i}\theta}}{2}\Lambda_{21}\right)\\ &+ z\left(\frac{\delta\,\mathrm{e}^{-\mathrm{i}\phi}}{3}(\Lambda_{41} - \Gamma_{03}) - \frac{\beta\,\mathrm{e}^{-\mathrm{i}\theta}}{2}(\Lambda_{31} - \Gamma_{02}) + \frac{\delta\,\mathrm{e}^{\mathrm{i}\phi}}{3}(\Lambda_{14} - \Lambda_{03})\right.\\ &\left. - \frac{\beta\,\mathrm{e}^{\mathrm{i}\theta}}{2}(\Lambda_{13} - \Lambda_{02})\right) + \frac{p^2}{4}\left(\frac{\delta\,\mathrm{e}^{-\mathrm{i}\phi}}{4}(\Lambda_{31} - \Gamma_{02}) - \frac{\beta\,\mathrm{e}^{-\mathrm{i}\theta}}{2}(\Lambda_{21} - \Gamma_{01})\right.\\ &\left. + \frac{\delta\,\mathrm{e}^{\mathrm{i}\phi}}{4}(\Lambda_{13} - \Lambda_{02}) - \frac{\beta\,\mathrm{e}^{\mathrm{i}\theta}}{2}(\Lambda_{12} - \Lambda_{01})\right).\end{aligned} \tag{89}$$

Again, according to (82), we have the same expression for the mean value of P^2, but taking the negative of the real part in place of the real part.

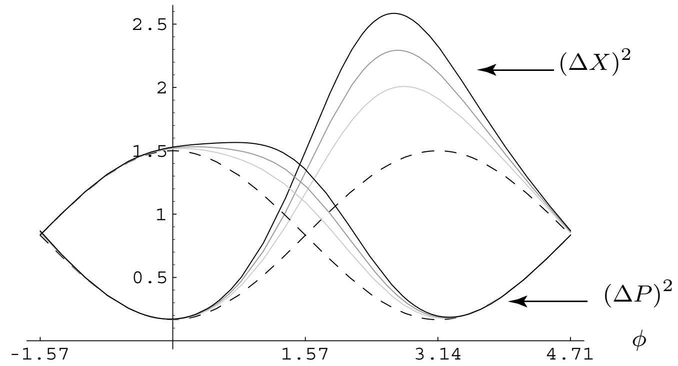

Figure 1. Graphs of the dispersions of *X* and *P* as functions of ϕ for $p = 0$ and $z = 0.000$, 0.0010, 0.0015, 0.0020.

Combining (88) with (89), according to equation (84), we get the dispersion of *X*. In the same way, we can obtain the dispersion of *P*. Inserting the matrix elements Γ_{ij} and Λ_{ij}, as given in appendix B, we can compute these dispersions explicitly.

Figure 1 shows the dispersions of *X* and *P* in the minimum uncertainty squeezed states in dashed lines, and in the deformed squeezed states in solid lines, as a function of ϕ for fixed values of the parameters δ, β, θ and p ($\delta = 0.5, \beta = 2.0, \theta = 0.8\pi, p = 0.00$) and for special values of $z = 0.0010, 0.0015, 0.0020$ (from the smaller to the greater grey level). We observe that, as a consequence of the small deformations in the parameters z the squeezing properties of *X* and *P* have not been essentially changed. Thus, in all the cases, we have *P*-squeezed states when $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, and *X*-squeezed states when $\frac{\pi}{2} < \phi < \frac{3\pi}{2}$. Also we observe that the product of the dispersions of *X* and *P* in the deformed squeezed states, for a given value of ϕ, is always greater than the product of the dispersions in the minimum uncertainty states, as required by the SRUR. This difference is more remarkable for values of ϕ in the range $\frac{\pi}{2} \leqslant \phi < \frac{3\pi}{2}$. Let us note that when $\phi = \pm\frac{\pi}{2}$, the MUS are coherent states, in the sense of the SRUR, i,e., the dispersion of *X* and *P* is the same. Indeed, in all these cases, $(\Delta X)_0{}^2 = (\Delta P)_0{}^2 = 0.83$. This value is conserved by the product of the dispersions of *X* and *P* in the deformed squeezed states when $\phi = -\frac{\pi}{2}$, but when $\phi = \frac{\pi}{2}$, it grows quickly as z increases.

Figure 2 shows the dispersions of *X* and *P* in the minimum uncertainty squeezed states in dashed lines, and in the deformed squeezed states in solid lines, as a function of ϕ for fixed values of the parameters δ, β, θ and z ($\delta = 0.5, \beta = 2.0, \theta = 0.8\pi, z = 0.0030$) and for special values of $p = 0.00, 0.06, 0.11$ (from the greater to the smaller grey level). We observe that the product of dispersions of *X* and *P* decreases when p increases. Thus the influence of the p parameter on the first order in z deformed states is to reduce the uncertainty product of *X* and *P* and to bring closer this quantity to the minimum uncertainty values.

Figure 3 shows the typical behaviour of the dispersions of *X* and *P* in the minimum uncertainty squeezed states in dashed lines, and in the deformed squeezed states in solid lines, as a function of δ for $\phi = 0.5, \beta = 2.0, \theta = 0.8\pi, z = 0.0025$ and $p = 0.001$. We observe again that, as a consequence of the small deformations in z and p, the squeezing properties of *X* and *P* have not been essentially changed. Thus, the figure shows the behaviour of *P*-squeezed and *P*-deformed squeezed states. When $0 < \delta \lesssim 0.75$, the product of the dispersions of *X* and *P*, in the deformed squeezed states is always greater than the corresponding product in the minimum uncertainty squeezed states, as required by the SRUR. For higher values of δ,

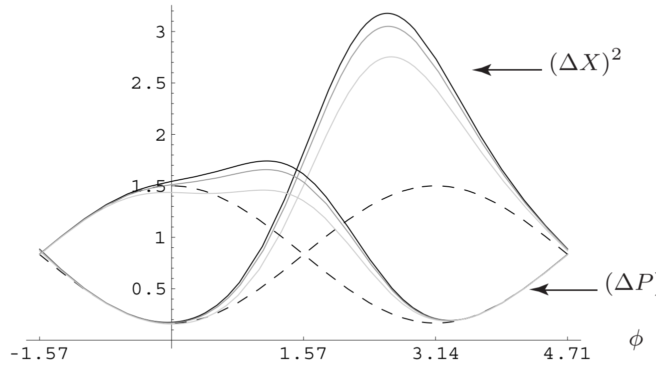

Figure 2. Graphs of the dispersions of X and P as functions of ϕ for $z = 0.0030$, $p = 0.00, 0.06, 0.11$, $\beta = 2.0$, $\theta = 0.8\pi$ and $\delta = 0.5$.

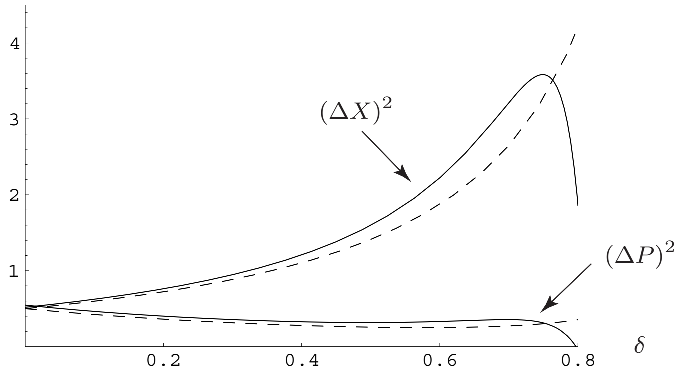

Figure 3. Graphs of the dispersions of X and P as functions of δ for $z = 0.0025$, $p = 0.01$, $\beta = 2.0$, $\theta = 0.8\pi$ and $\phi = \frac{\pi}{6}$.

only the dashed lines represent the true behaviour of the dispersions of X and P. Indeed, the approximation for the deformed squeezed states, in this region, is not valid. These states are no longer normalizable.

4.2. *General formulae for the dispersions of X and P in the z deformed states*

The mean values of X^k, $k = 1, 2, \ldots$, in states (45) can be expressed in the form

$$\langle\varphi|X^k|\varphi\rangle = \frac{\frac{\partial^k}{\partial\tau^k}\langle\widetilde{\varphi}|\mathrm{e}^{\tau X}|\widetilde{\varphi}\rangle|_{\tau=0}}{\langle\widetilde{\varphi}|\widetilde{\varphi}\rangle} = \frac{\frac{\partial^k}{\partial\tau^k}\left\{\mathrm{e}^{-\frac{\tau^2}{4}}\langle\widetilde{\varphi}|\mathrm{e}^{\frac{\tau}{\sqrt{2}}a}\,\mathrm{e}^{\frac{\tau}{\sqrt{2}}a^\dagger}|\widetilde{\varphi}\rangle\right\}\Big|_{\tau=0}}{\langle\widetilde{\varphi}|\widetilde{\varphi}\rangle}, \tag{90}$$

where

$$|\widetilde{\varphi}\rangle = \exp\left(\mathrm{e}^{-za^\dagger}\frac{(\mu - \lambda z + \mu z a^\dagger)}{z^2}\right)|0\rangle. \tag{91}$$

Inserting these results into (84) and evaluating we get

$$(\Delta X)^2 = -\frac{1}{2} + \frac{\frac{\partial^2}{\partial\tau^2}\langle\widetilde{\varphi}|\mathrm{e}^{\frac{\tau}{\sqrt{2}}a}\mathrm{e}^{\frac{\tau}{\sqrt{2}}a^\dagger}|\widetilde{\varphi}\rangle\big|_{\tau=0}}{\langle\widetilde{\varphi}|\widetilde{\varphi}\rangle} - \left(\frac{\frac{\partial}{\partial\tau}\langle\widetilde{\varphi}|\mathrm{e}^{\frac{\tau}{\sqrt{2}}a}\,\mathrm{e}^{\frac{\tau}{\sqrt{2}}a^\dagger}|\widetilde{\varphi}\rangle\big|_{\tau=0}}{\langle\widetilde{\varphi}|\widetilde{\varphi}\rangle}\right)^2. \tag{92}$$

To compute the matrix element $\langle\widetilde{\varphi}|\mathrm{e}^{\frac{\tau}{\sqrt{2}}a}\,\mathrm{e}^{\frac{\tau}{\sqrt{2}}a^\dagger}|\widetilde{\varphi}\rangle$, we can firstly write

$$\mathrm{e}^{\frac{\tau}{\sqrt{2}}a^\dagger}|\widetilde{\varphi}\rangle = \sum_{n=0}^{\infty} C_n(\tau)|n\rangle \tag{93}$$

and then compute the coefficients $C_n(\tau), n = 0, 1, 2, \ldots,$ in the Bargman representation, in the same way as we have done in section 3.1.1. That is

$$\langle\widetilde{\varphi}|\mathrm{e}^{\frac{\tau}{\sqrt{2}}a}\,\mathrm{e}^{\frac{\tau}{\sqrt{2}}a^\dagger}|\widetilde{\varphi}\rangle = \sum_{n=0}^{\infty} \bar{C}_n(\tau)C_n(\tau), \tag{94}$$

where

$$C_n(\tau) = \frac{1}{\sqrt{n!}}\sum_{r=0}^{n}\binom{n}{r}\left(\frac{\tau}{\sqrt{2}}\right)^r z^{n-r}\sum_{k=0}^{\infty}\sum_{m=0}^{\tilde{k}_<}\binom{n-r}{m}\frac{(-k)^{n-r-m}}{(k-m)!}\left(\frac{\mu}{z^2}\right)^m\left(\frac{\mu}{z^2}-\frac{\lambda}{z}\right)^{k-m}, \tag{95}$$

with $\tilde{k}_<$ the minimum between k and $n-r$.

Inserting (94) into (92) and evaluating again we get

$$(\Delta X)^2 = -\frac{1}{2} + \frac{\sum_{n=0}^{\infty}[\bar{C}_n(\tau)C_n''(\tau) + \bar{C}_n''C_n(\tau) + 2\bar{C}_n'C_n'(\tau)]|_{\tau=0}}{\sum_{n=0}^{\infty}\bar{C}_n(0)C_n(0)} - \left(\frac{\sum_{n=0}^{\infty}[\bar{C}_n(\tau)C_n'(\tau) + \bar{C}_n'(\tau)C_n(\tau)]|_{\tau=0}}{\sum_{n=0}^{\infty}\bar{C}_n(0)C_n(0)}\right)^2, \tag{96}$$

where, for instance, $C_n'(\tau) = \frac{\mathrm{d}C_n}{\mathrm{d}\tau}(\tau)$. From (95), we obtain

$$C_n'(0) = \frac{1}{\sqrt{n!}}\binom{n}{1}\frac{1}{\sqrt{2}}z^{n-1}\sum_{k=0}^{\infty}\sum_{m=0}^{\tilde{k}_1}\binom{n-1}{m}\frac{(-k)^{n-1-m}}{(k-m)!}\left(\frac{\mu}{z^2}\right)^m\left(\frac{\mu}{z^2}-\frac{\lambda}{z}\right)^{k-m}, \tag{97}$$

when $n = 1, 2, \ldots,$ with $\tilde{k}_1$ the minimum between k and $n-1$,

$$C_n''(0) = \frac{1}{\sqrt{n!}}\binom{n}{2}z^{n-2}\sum_{k=0}^{\infty}\sum_{m=0}^{\tilde{k}_2}\binom{n-2}{m}\frac{(-k)^{n-2-m}}{(k-m)!}\left(\frac{\mu}{z^2}\right)^m\left(\frac{\mu}{z^2}-\frac{\lambda}{z}\right)^{k-m}, \tag{98}$$

when $n = 2, 3, \ldots,$ with $\tilde{k}_2$ the minimum between k and $n-2$, and

$$C_0'(0) = C_0''(0) = C_1''(0) = 0. \tag{99}$$

The formula to the dispersion of P can be obtained from (96) changing the τ argument of $C_n(\tau)$ by $\mathrm{i}\tau$ and then deriving and evaluating $\tau = 0$. Thus, dispersions formulae of X and P at all order in z can be obtained. The first-order perturbation formulae of these dispersions must correspond to the dispersions obtained in the preceding subsection, in the limit when p goes to zero.

4.3. η-pseudo Hermitian and Hermitian Hamiltonians

In this section we show that the subset of deformed coherent states (46), corresponding to the eigenvalue $\lambda = 0$, are the coherent states associated with an η-pseudo Hermitian Hamiltonian [19] but also, up with a similarity transformation, the coherent states associated with a Hermitian Hamiltonian, both isospectral to the harmonic oscillator Hamiltonian. Indeed, when $\lambda = 0$, eigenstates (46) correspond to the solutions of the eigenvalue equation

$$\mathcal{A}|\psi\rangle = -\nu|\psi\rangle, \qquad \nu \in \mathbb{C}, \tag{100}$$

where $\mathcal{A} = a + \mu a^\dagger \, \mathrm{e}^{-za^\dagger}$. These solutions can be written in the form

$$|\psi; -\nu\rangle = \tilde{N}_0(\mu, -\nu, z) G(\mu, z) \, \mathrm{e}^{-\nu a^\dagger} |0\rangle, \tag{101}$$

where

$$G(\mu, z) = \exp\left(-\mu \sum_{k=0}^{\infty} \frac{(-za^\dagger)^k}{k!} \frac{(a^\dagger)^2}{(k+2)}\right), \tag{102}$$

and $\tilde{N}_0(\mu, -\nu, z)$ is a normalization constant.

Let us now define the operator

$$\mathcal{H} = G a^\dagger a G^{-1}, \tag{103}$$

which satisfies

$$\mathcal{H}^\dagger = \eta \mathcal{H} \eta^{-1}, \tag{104}$$

where η is the Hermitian operator

$$\eta(\mu, z) = (G^{-1})^\dagger G^{-1}. \tag{105}$$

Thus $\mathcal{H}$ is an η-pseudo Hermitian Hamiltonian [19]. Moreover, as

$$G a^\dagger G^{-1} = a^\dagger, \qquad G a G^{-1} = \mathcal{A}, \tag{106}$$

we get

$$\mathcal{H} = a^\dagger \mathcal{A} = a^\dagger \big(a + \mu a^\dagger \, \mathrm{e}^{-za^\dagger}\big) = a^\dagger a + \mu \, \mathrm{e}^{-za^\dagger} (a^\dagger)^2. \tag{107}$$

On the other hand, by construction, it is easy to verify that

$$[\mathcal{H}, \mathcal{A}] = -\mathcal{A}, \qquad [\mathcal{H}, a^\dagger] = a^\dagger, \qquad [\mathcal{A}, a^\dagger] = 1 \tag{108}$$

and

$$\mathcal{H}|E_0\rangle = 0, \tag{109}$$

where

$$|E_0\rangle = \tilde{N}_0(\mu, 0, z) G(\mu, z)|0\rangle. \tag{110}$$

This state is thus an eigenstate of $\mathcal{A}$ corresponding to the eigenvalue $\nu = 0$. Thus, according to (108) and (109), the Hamiltonian $\mathcal{H}$ is isospectral to the harmonic oscillator Hamiltonian. $\mathcal{A}$ represents an annihilation operator for this system and their eigenstates (101) are the associated coherent states of $\mathcal{H}$.

Let us mention that $\mathcal{H}$ verifies all the useful properties of pseudo-Hermitian operators [31]. For instance,$\mathcal{H}$ is Hermitian on the physical Hilbert space $\mathfrak{H}_{\mathrm{phys}}$ spanned by their corresponding eigenstates $|\psi_n\rangle \propto (a^\dagger)^n G|0\rangle, n = 0, 1, 2, \ldots,$ endowed with the positive-definite inner product $\langle \cdot | \eta \, \cdot \rangle$. Also, $\mathcal{H}$ may be mapped to a Hermitian Hamiltonian $\tilde{\mathcal{H}}$ by a similarity transformation $\tilde{\mathcal{H}} = \hat{\rho} \mathcal{H} \hat{\rho}^{-1}$, where $\hat{\rho}(\mu, z) = \sqrt{\eta(\mu, z)} = \sqrt{G^{-1\dagger} G^{-1}}$, is a Hermitian operator on a Hilbert space $\mathfrak{H}$ formed of same vectorial space $\mathfrak{H}_{\mathrm{phys}}$ but endowed with the original inner product $\langle \cdot | \cdot \rangle$. Thus, in our case, according to (103), the Hermitian Hamiltonian $\tilde{\mathcal{H}}$ is unitarily equivalent to the standard harmonic oscillator Hamiltonian and is given by

$$\tilde{\mathcal{H}} = \hat{\rho} G a^\dagger a G^{-1} \hat{\rho}^{-1}. \tag{111}$$

Indeed,

$$\hat{\rho} G (\hat{\rho} G)^\dagger = \hat{\rho} G G^\dagger \hat{\rho}^\dagger = \hat{\rho} \eta^{-1} \hat{\rho} = \hat{\rho} (\hat{\rho}^2)^{-1} \hat{\rho} = I \tag{112}$$

and

$$(\hat{\rho}G)^{\dagger}\hat{\rho}G = G^{\dagger}\hat{\rho}^{\dagger}\hat{\rho}G = G^{\dagger}\hat{\rho}^{2}G = G^{\dagger}(G^{-1})^{\dagger}G^{-1}G = I, \tag{113}$$

that is $(\hat{\rho}G)^{\dagger} = (\hat{\rho}G)^{-1}$, i.e., $\hat{\rho}G$ is an unitary operator.

Let us note that in the absence of deformation ($z = 0$) the operator $\hat{\rho}$ is given by

$$\hat{\rho}(\mu, 0) = \sqrt{\exp\left(\bar{\mu}\frac{a^2}{2}\right)\exp\left(\mu\frac{a^{\dagger 2}}{2}\right)} = \left[\exp\left(\int_0^1 [\bar{\mu}K_- + \mu K_+ + \varsigma(s)K_3]\,\mathrm{d}s\right)\right]^{\frac{1}{2}}, \tag{114}$$

where $K_- = \frac{a^2}{2}$, $K_+ = \frac{(a^{\dagger})^2}{2}$ and $K_3 = \frac{1}{4}(aa^{\dagger} + a^{\dagger}a)$ are the standard bosonic realizations of the $su(1,1)$ Lie algebra generators verifying the commutation relations

$$[K_-, K_+] = 2K_3, \qquad [K_3, K_{\pm}] = \pm K_{\pm} \tag{115}$$

and

$$\varsigma(s) = -2\frac{\mathrm{d}}{\mathrm{d}s}\ln q(s), \tag{116}$$

where

$$q(s) = \cosh(|\mu|(1-s)) + |\mu|\sinh(|\mu|(1-s)). \tag{117}$$

In this case, Hamiltonian (111) becomes

$$\tilde{\mathcal{H}} = \hat{\rho}(\mu, 0)[a^{\dagger}a + \mu(a^{\dagger})^2]\hat{\rho}^{-1}(\mu, 0), \tag{118}$$

and represents a Hermitian Hamiltonian describing two photon processes in a single mode. To know the explicit form of this Hamiltonian we must firstly factorize operator (114) in the form of a product of exponential operators of each $su(1,1)$ generators and then insert it into (118). This process requires to solve some Ricatti-type differential equations.

For small values of z, Hamiltonian (111) describes corrections to the energy of this system as a consequence of the deformation. In general, when $z \neq 0$, Hamiltonian (111) represents multi-photon processes in a single mode.

The generalized coherent states associated with the system described by (111), can be easily obtained from the coherent states associated with the standard harmonic oscillator. Indeed, they are given by

$$|\nu, z, \mu\rangle = \hat{\rho}(\mu, z)G(\mu, z)D(\nu)|0\rangle, \tag{119}$$

where $D(\nu)$ is the standard unitary displacement operator defined at the end of subsection 3.1.2. These coherent states correspond to the coherent states associated with the pseudo-Hermitian Hamiltonian (107), up to the transformation $\hat{\rho}(\mu, z)$, and are eigenstates of the annihilation operator $\tilde{\mathcal{A}} = \hat{\rho}(\mu, z)\mathcal{A}\hat{\rho}^{-1}(\mu, z)$ corresponding to the eigenvalue ν.

5. Conclusions

In this paper, we have found some realizations of the deformed quantum Heisenberg Lie algebra $\mathcal{U}_{z,p}(h(2))$, in terms of the usual creation and annihilation operators associated with the Fock space representation of the standard harmonic oscillator. The method used to get these realizations can be easily applied to find the realizations of other quantum Hopf algebras and super-algebras, such as the bosonic and fermionic oscillators Hopf algebras [32] or the quantum super-Heisenberg algebra, that can also be obtained by using the R-matrix approach.

We have computed the AES associated with $\mathcal{U}_{z,p}(h(2))$. We have seen that the set of AES contains the set of coherent and squeezed states associated with the standard harmonic oscillator system but also a new class of deformed coherent and squeezed states, parametrized by the deformation parameters. We have studied the behaviour of the dispersions of the position and linear momentum operators of a particle in a class of perturbed squeezed states and we have compared them with the behaviour of these dispersions in the minimum uncertainty squeezed states. Also we have computed these dispersions on the deformed states associated with $\mathcal{U}_{z,0}(h(2))$ for all values of the z parameter. To first order in z, these last dispersions reduce to the perturbed ones obtained in $\mathcal{U}_{z,p}(h(2))$, when p goes to zero. Besides, we have constructed a η-pseudo Hermitian Hamiltonian [19] to which a subset of the set of algebra eigenstates associated with $\mathcal{U}_{z,0}(h(2))$ are the coherent states. From this point of view, our deformed states are linked to Hamiltonians presenting important physical aspects [31]. Indeed, our pseudo-Hermitian Hamiltonian verifies naturally all the properties of pseudo-Hermitian Hamiltonians such as the existence of associated biorthonormal basis, resolution of the identity, positive-definite inner product, physical Hilbert space, unitary and invertible operators mapping the pseudo-Hermitian operators to the Hermitian ones, etc. Thus, with the help of pseudo-Hermitian quantum mechanics techniques we are allowed to compute, for instance, the spectrum, the eigenstates and the associated coherent states of complicated deformed Hermitian Hamiltonians describing multi-photon processes in a single mode. Also, we can compute more easily quantities such as mean values of physical observables and transition amplitudes. Moreover, it could be interesting to know, at least for small values of the deformation parameter z, the explicit form of the resolution of the identity verified by the generalized coherent states (119). Indeed, this fact could have important consequences, for instance, in the study of corrections to the time evolution of the quantum fluctuations associated with the quadratures of the position and linear momentum of a system characterized by a Hamiltonian describing one and two photon processes in a single mode [33]. This is a non-trivial problem and it could be developed elsewhere.

On the other hand, we have found new classes of deformed squeezed states, parametrized by a real paragrassmann number, i.e., a number z such that $z^{k_0} = 0$, for some $k_0 \in \mathbb{N}$. These states can be normalized, even if z is considered as a complex paragrassmann number. In this last case, when $k_0 = 2$, we should interpret z as an odd complex Grassmann number and compare this new classes of deformed squeezed states with those associated with the η-super-pseudo-Hermitian Hamiltonians [15].

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Appendix A. Solving a paragrassmann valued differential equation

In this appendix we are interested in solving the differential equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}\xi} + (\mu\xi + \nu)\sum_{l=0}^{k_0-1}\frac{(-z)^l}{l!}\frac{\mathrm{d}^l}{\mathrm{d}\xi^l}\right]\varphi(\xi) = \lambda\varphi(\xi), \qquad \mu, \nu, \lambda \in \mathbb{C}, \tag{A.1}$$

where $k_0 \in \mathbb{N}, k_0 \geqslant 1$ and z is a paragrassmann generator such that $z^k = 0, \forall k \geqslant k_0$.

Let us assume a solution of the type

$$\varphi(\xi) = \sum_{k=0}^{k_0-1} z^k \varphi_k(\xi). \tag{A.2}$$

Inserting this solution into (A.1), we get

$$\left[\sum_{k=0}^{k_0-1} z^k \frac{\mathrm{d}\varphi_k}{\mathrm{d}\xi} + (\mu\xi + \nu)\sum_{l=0}^{k_0-1}\sum_{k=0}^{k_0-1} \frac{(-1)^l (z)^{k+l}}{l!}\frac{\mathrm{d}^l\varphi_k}{\mathrm{d}\xi^l}\right] = \lambda \sum_{k=0}^{k_0-1} z^k \varphi_k. \tag{A.3}$$

Identifying the coefficients of independent powers z^k, $k = 0, 1, 2, \ldots, k_0 - 1$, in this equality, we get the following system of differential equations ($k = 1, \ldots, k_0 - 1$):

$$\frac{\mathrm{d}\varphi_k}{\mathrm{d}\xi} + (\mu\xi + \nu)\sum_{l=1}^{k} \frac{(-1)^l}{l!}\frac{\mathrm{d}^l\varphi_{k-l}}{\mathrm{d}\xi^l} = [(\lambda - \nu) - \mu\xi]\,\varphi_k, \tag{A.4}$$

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}\xi} = [(\lambda - \nu) - \mu\xi]\,\varphi_0. \tag{A.5}$$

Let us note that we can solve this system of differential equations proceeding by iteration. Indeed, from equation (A.5), we get

$$\varphi_0(\xi) = C_0 \exp\left((\lambda - \nu)\xi - \tfrac{1}{2}\mu\xi^2\right), \tag{A.6}$$

where C_0 is an arbitrary integration constant. Also, from equation (A.4), for a given value of k, the general solution $\varphi_k(\xi)$ is of the type

$$\varphi_k(\xi) = [C_k + A_k(\xi)] \exp\left((\lambda - \nu)\xi - \tfrac{1}{2}\mu\xi^2\right), \qquad k = 1, \ldots, k_0 - 1, \tag{A.7}$$

where the C_k are arbitrary integration constants and $A_k(\xi)$ are functions of ξ which can be determined by solving the system of differential equations ($k = 1, 2, \ldots, k_0 - 1$)

$$\frac{\mathrm{d}A_k}{\mathrm{d}\xi} = \exp\left(\frac{1}{2}\mu\xi^2 - (\lambda - \nu)\xi\right) \times (\mu\xi + \nu)\sum_{l=1}^{k} \frac{(-1)^{l+1}}{l!}\frac{\mathrm{d}^l}{\mathrm{d}\xi^l}\left[(C_{k-l} + A_{k-l})\exp\left((\lambda - \nu)\xi - \frac{1}{2}\mu\xi^2\right)\right]. \tag{A.8}$$

Using the Leibnitz's derivation rule it is easy to prove that

$$\begin{aligned}\exp\left(\frac{1}{2}\mu\xi^2 - (\lambda - \nu)\xi\right)&\frac{\mathrm{d}^l}{\mathrm{d}\xi^l}\left[(C_{k-l} + A_{k-l})\exp\left((\lambda - \nu)\xi - \frac{1}{2}\mu\xi^2\right)\right] \\ &= \sum_{m=0}^{l}\binom{l}{m}\Bigg[C_{k-l}(\lambda - \nu)^{l-m}\left(\frac{\mu}{2}\right)^{m/2}(-1)^m H_m\left(\sqrt{\frac{\mu}{2}}\xi\right) \\ &\quad + \frac{\mathrm{d}^{l-m}A_{k-l}}{\mathrm{d}\xi^{l-m}}\sum_{s=0}^{m}\binom{m}{s}(\lambda - \nu)^{m-s}\left(\frac{\mu}{2}\right)^{s/2}(-1)^s H_s\left(\sqrt{\frac{\mu}{2}}\xi\right)\Bigg],\end{aligned} \tag{A.9}$$

where

$$H_m(x) = \mathrm{e}^{x^2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\mathrm{e}^{-x^2}, \qquad m = 0, 1, \ldots, \tag{A.10}$$

are the Hermite polynomials.

Inserting these results into (A.8) and integrating with respect to ξ, we get

$$\begin{aligned}A_k(\xi) = \sum_{l=1}^{k}\sum_{m=0}^{l}\frac{(-1)^{l+1}}{l!}\binom{l}{m}\int(\mu\xi + \nu)\Bigg[C_{k-l}(\lambda - \nu)^{l-m}\left(\frac{\mu}{2}\right)^{m/2}(-1)^m H_m\left(\sqrt{\frac{\mu}{2}}\xi\right) \\ + \frac{\mathrm{d}^{l-m}A_{k-l}}{\mathrm{d}\xi^{l-m}}\sum_{s=0}^{m}\binom{m}{s}(\lambda - \nu)^{m-s}\left(\frac{\mu}{2}\right)^{s/2}(-1)^s H_s\left(\sqrt{\frac{\mu}{2}}\xi\right)\Bigg]\,\mathrm{d}\xi,\end{aligned} \tag{A.11}$$

when $k = 1, 2, \ldots, k_0 - 1$. This system of integral equations can be solved by iteration using the initial condition $A_0(\xi) = 0$. For instance, when $k_0 \geqslant 2$, from equation (A.11), we get

$$A_1(\xi) = \left[(\lambda - \nu)\nu\xi + \mu(\lambda - 2\nu)\frac{\xi^2}{2} - \mu^2\frac{\xi^3}{3}\right]C_0. \tag{A.12}$$

When $k_0 \geqslant 3$, from (A.11), we get

$$\begin{aligned} A_2(\xi) = \Bigg[& \left(\frac{\lambda^2\nu}{2} + \frac{\mu\nu}{2} + 2\lambda\nu^2 - \frac{3\nu^3}{2}\right)\xi \\ & - \left(\frac{\lambda^2\mu}{4} - \frac{\mu^2}{4} - 2\lambda\mu\nu - \frac{\lambda^2\nu^2}{2} + \frac{9\mu\nu^2}{4} + \lambda\nu^3 - \frac{\nu^4}{2}\right)\xi^2 \\ & + \left(\frac{2\lambda\mu^2}{3} + \frac{\lambda^2\mu\nu}{2} - \frac{3\mu^2\nu}{2} - \frac{3\lambda\mu\nu^2}{2} + \mu\nu^3\right)\xi^3 \\ & + \left(\frac{\lambda^2\mu^2}{8} - \frac{3\mu^3}{8} - \frac{5\lambda\mu^2\nu}{6} + \frac{5\mu^2\nu^2}{6}\right)\xi^4 \\ & - \left(\frac{\lambda\mu^3}{6} - \frac{\mu^3\nu}{3}\right)\xi^5 + \frac{\mu^4\xi^6}{18}\Bigg]C_0 \\ & + \left((\lambda - \nu)\nu\xi + \mu(\lambda - 2\nu)\frac{\xi^2}{2} - \mu^2\frac{\xi^3}{3}\right)C_1. \end{aligned} \tag{A.13}$$

Finally, the general solution of the differential equation system (A.1) is obtained by inserting (A.7) into (A.2)

$$\varphi(\xi) = \left[\sum_{k=0}^{k_0-1} z^k(C_k + A_k(\xi))\right]\exp\left((\lambda - \nu)\xi - \frac{1}{2}\mu\xi^2\right), \tag{A.14}$$

with $A_k(\xi)$ given in equation (A.11). We note that there exists an independent solution for each integration constant C_k, $k = 0, 1, \ldots, k_0 - 1$.

In the case $\nu = 0$, equation (A.11) reduces to ($k = 1, 2, \ldots, k_0 - 1$)

$$\begin{aligned} A_k(\xi) = \mu\sum_{l=1}^{k}\sum_{m=0}^{l}\frac{(-1)^{l+1}}{l!}\binom{l}{m}\int\xi\Bigg[& C_{k-l}\lambda^{l-m}\left(\frac{\mu}{2}\right)^{m/2}(-1)^m H_m\left(\sqrt{\frac{\mu}{2}}\xi\right) \\ & + \frac{\mathrm{d}^{l-m}A_{k-l}}{\mathrm{d}\xi^{l-m}}\sum_{s=0}^{m}\binom{m}{s}\lambda^{m-s}\left(\frac{\mu}{2}\right)^{s/2}(-1)^s H_s\left(\sqrt{\frac{\mu}{2}}\xi\right)\Bigg]\mathrm{d}\xi. \end{aligned} \tag{A.15}$$

Thus, for instance, from equation (A.15), when $k_0 \geqslant 2$, we get

$$A_1(\xi) = \mu\left(\lambda\frac{\xi^2}{2} - \mu\frac{\xi^3}{3}\right)C_0. \tag{A.16}$$

When $k_0 \geqslant 3$, we get

$$\begin{aligned} A_2(\xi) = & \left[\left(\frac{\mu^2}{4} - \frac{\lambda^2\mu}{4}\right)\xi^2 + \frac{2\lambda\mu^2}{3}\xi^3 + \left(\frac{\lambda^2\mu^2}{8} - \frac{3\mu^3}{8}\right)\xi^4 - \frac{\lambda\mu^3}{6}\xi^5 + \frac{\mu^4\xi^6}{18}\right]C_0 \\ & + \left(\frac{\mu\lambda}{2}\xi^2 - \frac{\mu^2}{3}\xi^3\right)C_1. \end{aligned} \tag{A.17}$$

In the case $\mu = 0$, equation (A.11) reduces to ($k = 1, 2, \ldots, k_0 - 1$)

$$A_k(\xi) = \nu \sum_{l=1}^{k} \frac{(-1)^{l+1}}{l!} \left[(\lambda - \nu)^l \left(\xi C_{k-l} + \int A_{k-l}\, \mathrm{d}\xi \right) + \sum_{m=0}^{l-1} \binom{l}{m} (\lambda - \nu)^m \frac{\mathrm{d}^{l-1-m}}{\mathrm{d}\xi^{l-1-m}} A_{k-l} \right]. \tag{A.18}$$

For instance, from this last equation, when $k_0 \geqslant 2$, we get

$$A_1(\xi) = (\lambda - \nu)\nu\xi C_0 \tag{A.19}$$

and when $k_0 \geqslant 3$, we get

$$A_2(\xi) = \left[\left(\frac{\lambda^2\nu}{2} + 2\lambda\nu^2 - \frac{3\nu^3}{2} \right) \xi + \left(\frac{\lambda^2\nu^2}{2} - \lambda\nu^3 + \frac{\nu^4}{2} \right) \xi^2 \right] C_0 + (\lambda - \nu)\nu\xi C_1. \tag{A.20}$$

Appendix B. Matrix elements

In section 4.1, we need to compute the following matrix elements:

$$\Gamma_{kl} = \langle 0| D^\dagger \left(\frac{\beta\, \mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}} \right) S^\dagger(-\tan^{-1}(\delta)\, \mathrm{e}^{\mathrm{i}\phi}) a^{\dagger k} a^l S(-\tan^{-1}(\delta)\, \mathrm{e}^{\mathrm{i}\phi}) D \left(\frac{\beta\, \mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}} \right) |0\rangle, \tag{B.1}$$

and

$$\Lambda_{kl} = \langle 0| D^\dagger \left(\frac{\beta\, \mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}} \right) S^\dagger(-\tan^{-1}(\delta)\, \mathrm{e}^{\mathrm{i}\phi}) a^k a^{\dagger l} S(-\tan^{-1}(\delta)\, \mathrm{e}^{\mathrm{i}\phi}) D \left(\frac{\beta\, \mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}} \right) |0\rangle, \tag{B.2}$$

with $k, l = 0, 1, 2, \ldots$. Using the relation

$$S^\dagger(-\tan^{-1}(\delta)\, \mathrm{e}^{\mathrm{i}\phi}) a S(-\tan^{-1}(\delta)\, \mathrm{e}^{\mathrm{i}\phi}) = \frac{1}{\sqrt{1-\delta^2}} (a - \delta\, \mathrm{e}^{\mathrm{i}\phi} a^\dagger), \tag{B.3}$$

we can write them in the form

$$\Gamma_{kl} = \langle 0| D^\dagger \left(\frac{\beta\, \mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}} \right) \frac{(a^\dagger - \delta\, \mathrm{e}^{-\mathrm{i}\phi} a)^k (a - \delta\, \mathrm{e}^{\mathrm{i}\phi} a^\dagger)^l}{(1-\delta^2)^{\frac{k+l}{2}}} D \left(\frac{\beta\, \mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}} \right) |0\rangle \tag{B.4}$$

and

$$\Lambda_{kl} = \langle 0| D^\dagger \left(\frac{\beta\, \mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}} \right) \frac{(a - \delta\, \mathrm{e}^{\mathrm{i}\phi} a^\dagger)^k (a^\dagger - \delta\, \mathrm{e}^{-\mathrm{i}\phi} a)^l}{(1-\delta^2)^{\frac{k+l}{2}}} D \left(\frac{\beta\, \mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}} \right) |0\rangle, \tag{B.5}$$

respectively. From the above expressions, it is clear that

$$\Gamma_{0l} = \bar{\Gamma}_{l0} = \Lambda_{l0} = \bar{\Lambda}_{0l}, \qquad \Gamma_{ll} = \bar{\Gamma}_{ll}, \qquad \Lambda_{ll} = \bar{\Lambda}_{ll}, \qquad l = 0, 1, \ldots, \tag{B.6}$$

and

$$\Gamma_{kl} = \bar{\Gamma}_{lk}, \qquad \Lambda_{kl} = \bar{\Lambda}_{lk}, \quad k, l = 0, 1, \ldots. \tag{B.7}$$

We note that the Γ_{kl} matrix elements correspond to

$$\frac{\partial^k}{\partial\sigma^k} \frac{\partial^l}{\partial\tau^l} \langle 0| D^\dagger \left(\frac{\beta\, \mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}} \right) \frac{\exp[\sigma(a^\dagger - \delta\, \mathrm{e}^{-\mathrm{i}\phi} a)]}{(1-\delta^2)^{k/2}} \frac{\exp[\tau(a - \delta\, \mathrm{e}^{\mathrm{i}\phi} a^\dagger)]}{(1-\delta^2)^{l/2}} D \left(\frac{\beta\, \mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}} \right) |0\rangle, \tag{B.8}$$

when σ and τ go to zero. Applying the usual B H C formula to disentangle the exponentials factors, we get

$$\exp[\sigma(a^\dagger - \delta\,\mathrm{e}^{-\mathrm{i}\phi}a)]\exp[\tau(a - \delta\,\mathrm{e}^{\mathrm{i}\phi}a^\dagger)] = \exp\left[\sigma\tau\delta^2 - \tfrac{1}{2}\sigma^2\delta\,\mathrm{e}^{-\mathrm{i}\phi} - \tfrac{1}{2}\tau^2\delta\,\mathrm{e}^{\mathrm{i}\phi}\right] \times \exp[(\sigma - \tau\delta\,\mathrm{e}^{\mathrm{i}\phi})a^\dagger]\exp[(\tau - \sigma\delta\,\mathrm{e}^{-\mathrm{i}\phi})a]. \tag{B.9}$$

Inserting this result in (B.8), and acting with the exponential operators on the coherent states, we get

$$\Gamma_{kl} = \frac{1}{(\sqrt{1-\delta^2})^{k+l}}\frac{\partial^k}{\partial\sigma^k}\frac{\partial^l}{\partial\tau^l}\left\{\exp\left[\sigma\tau\delta^2 - \frac{1}{2}\sigma^2\delta\,\mathrm{e}^{-\mathrm{i}\phi} - \frac{1}{2}\tau^2\delta\,\mathrm{e}^{\mathrm{i}\phi}\right] \times \exp\left[(\sigma - \tau\delta\,\mathrm{e}^{\mathrm{i}\phi})\frac{\beta\,\mathrm{e}^{-\mathrm{i}\theta}}{\sqrt{1-\delta^2}}\right]\exp\left[(\tau - \sigma\delta\,\mathrm{e}^{-\mathrm{i}\phi})\frac{\beta\,\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}}\right]\right\}\Bigg|_{\sigma=\tau=0}. \tag{B.10}$$

The matrix elements Λ_{kl} can be obtained in the same way. We get

$$\Lambda_{kl} = \frac{1}{(\sqrt{1-\delta^2})^{k+l}}\frac{\partial^k}{\partial\sigma^k}\frac{\partial^l}{\partial\tau^l}\left\{\exp\left[\sigma\tau - \frac{1}{2}\sigma^2\delta\,\mathrm{e}^{\mathrm{i}\phi} - \frac{1}{2}\tau^2\delta\,\mathrm{e}^{-\mathrm{i}\phi}\right] \times \exp\left[(\sigma - \tau\delta\,\mathrm{e}^{-\mathrm{i}\phi})\frac{\beta\,\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{1-\delta^2}}\right]\exp\left[(\tau - \sigma\delta\,\mathrm{e}^{\mathrm{i}\phi})\frac{\beta\,\mathrm{e}^{-\mathrm{i}\theta}}{\sqrt{1-\delta^2}}\right]\right\}\Bigg|_{\sigma=\tau=0}. \tag{B.11}$$

For example,

$$\Gamma_{00} = \Lambda_{00} = 1, \qquad \Gamma_{01} = \bar{\Lambda}_{01} = \frac{\beta\,\mathrm{e}^{\mathrm{i}\theta} - \beta\delta\,\mathrm{e}^{\mathrm{i}(\phi-\theta)}}{(1-\delta^2)},$$

$$\Gamma_{02} = \bar{\Lambda}_{02} = \frac{\beta^2\,\mathrm{e}^{2\mathrm{i}\theta} - \delta\,\mathrm{e}^{\mathrm{i}\phi}(2\beta^2 + 1 - \delta^2) + \beta^2\delta^2\,\mathrm{e}^{2\mathrm{i}(\phi-\theta)}}{(1-\delta^2)^2}$$

$$\Gamma_{11} = \Lambda_{11} - 1 = \frac{\beta^2(1+\delta^2) + \delta^2(1-\delta^2) - 2\beta^2\delta\cos(\phi-\delta)}{(1-\delta^2)^2},$$

$$\Lambda_{12} = [\beta\,\mathrm{e}^{-\mathrm{i}\theta}(\beta^2 + 2\beta^2\delta^2 + (2+\delta^2)(1-\delta^2)) - \beta\delta\,\mathrm{e}^{\mathrm{i}(\theta-\phi)}(2\beta^2 + \beta^2\delta^2 + 3(1-\delta^2)) + \beta^3\delta^2\,\mathrm{e}^{\mathrm{i}(3\theta-2\phi)} - \beta^3\delta\,\mathrm{e}^{\mathrm{i}(\phi-3\theta)}]/(1-\delta^2)^3. \tag{B.12}$$

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